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ABSTRACT

This essay considers recent research on classification as related to the study of mathematics in the early grades. The first section deals with the bases and types of classification used by children in free sorting activities. The next section specifically relates to elementary school mathematics. Experiments and theory on class inclusion, multiple classification, and relations and classification are summarized and discussed in detail. The author concludes that "while a great deal is known about classificational abilities of young children, more exploration is needed." (MM)

CLASSIFICATIONAL ABILITIES OF YOUNG CHILDREN

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CLASSIFICATIONAL ABILITIES OF YOUNG CHILDREN

Classification plays an important role in the study of mathematics from the beginning throughout to the most advanced stages. A few examples should suffice to establish the validity of this assertion. Following Klein it is possible to classify geometries according to properties left invariant under certain groups of linear transformation whereas in algebra, different categories of systems exist, i.e., groups, rings, etc. In the theory of relations, equivalence relations are distinct from relations of order, of which different types exist (linear orderings, partial orderings, etc.). Other, more detailed, examples from advanced mathematics could be given but are unnecessary. In all examples cited, the criteria of classification are purely intellectual constructions or inventions. That is not to say such criteria are completely arbitrary as they are inherent in mathematics. However, the study of mathematics in the early grades involves a different level of classification than does a study of advanced mathematics. No longer is the concern classification of mathematical objects but, rather, it is with classification of content of a universal nature. The focus of this paper is determined by classification implicit in the study of mathematics in the early grades.

On the Bases and Types of Classification

The reports of two studies were chosen for rather extensive discussion because they clarify, in a rather profound way, different

bases and types of classification of generic content employed by young children. Olver and Hornsby (1966) presented children from age six to nineteen with verbal materials and children from age six to eleven with pictorial material where the children were asked to tell how different objects were alike. In case of the verbal material, the children were presented with the words banana and peach visually and orally and were asked how "banana" and "peach" are alike. Then potato was added and questions concerning differences and likenesses were asked. The terms meat, milk, water, air, and germs were added to the list and treated analogously. At the end of the list, the term stove was presented and the experimenters asked only how stove was different than the others. A second list of words was also presented.

In case of the pictorial materials, 42 water color drawings were presented where each drawing represented familiar objects. The task was to select from these drawings pictures that were alike in some way. After selection, the child was asked to tell how the pictures he selected were alike. Each child was given ten trials.

In the analyses of data, two kinds of information were available; the basis on which collections were formed and a "type" of collection. In the first case, five main modes for equivalence were distinguished; a perceptible mode, a functional mode, an affective mode, a nominal mode, and fiat equivalence. The perceptible mode was based on qualities such as color, size, shape, or position. The functional mode was based on the use or function of the items. The affective mode was based on emotion aroused. The nominal mode was based on a name. Fiat equivalence was based on a statement that items were alike without giving further information even when prodded.

Different types of collections formed were superordinate collections, complexive collections, and thematic collections. Superordinate collections were based on common features characterizing the items. Complexive collections were more varied and included five varieties none of which tied all of the objects of a collection together. First was the case where contrasting or otherwise related properties were formed but the objects were not tied together by common attributes. For example, bell is black, horn is brown, etc. Second was the case where associative links were formed between neighboring items (edge matchings). Third was where all items were linked to a single item but not to each other (keyring). Fourth, two items were linked and other items were then tied to these two (associations). Fifth was the case where several subcollections were formed but not a general collection (multiple grouping). The last type of collection was termed thematic, and was characterized by collections formed by virtue of how the item fit into a story or sentence.

In case of the verbal presentations, the functional mode of classification was most dominant for all age groups with a steady increase with age; 49 percent of all responses at age six to 73 percent at age 19. About 25 percent of the responses of the six year olds were based on the perceptible mode with a decrease with age. In case of the type of collection formed, half of the groupings of the six year olds were complexive and half superordinate. By nine years of age, about three fourths were superordinate with an increase in formation of superordinate collections from that age on.

In the pictorial tasks, a greater incidence of the perceptible mode of classification occurred than occurred in the verbal tasks.

Forty-seven percent of the collections were formed using the perceptible mode by the six year olds and 27 percent by the eight year olds. More interesting data was that 61 percent of all collections formed by six year olds consisted of just a pair of objects, where other objects which could be classified with the two selected were ignored. Many of these pairs were formed using the thematic mode, such as "the bunny ate the carrots." In total, six year olds constructed 31 percent of their collections on the basis of such sentential structures and 38 percent on the basis of the complexive mode of classification in case of pictorially presented material.

In essence, the same pattern of growth emerged for both verbal and pictorial material. Classification for the six year olds reflected a basis in imagery both in the mode used as a basis for classifying and in how the collections were formed.

Parker and Halbrook (1969) conducted a study, the purpose of which was to ". . . investigate changes with age in the identification and combination of the common attributes of groups and to assess the influence of different attributes on this ability (p. 8)." Three different types of content was used; concrete, functional, and designative. The concrete content consisted of pictorial material with perceptually common features, such as form, color, or identity (some hats, etc.). This content was analogous to the perceptible mode of Olver and Hornsby. The functional content consisted of pictorial material where the common features were the use of the objects, analogous to the functional mode of Olver and Hornsby. The designative content consisted of pictorial material where the objects belonged to a class having a class name, such as animals, peas, fruit, etc., analogous to the nominal mode of Olver and Hornsby. The items presented

to the children were incomplete matrix items consisting of a row and a column with the common element missing. Six types of matrices were presented; concrete by concrete (CC), functional by functional (FF), designative by designative (DD), (CF), (CD), and (FD). Eighty children, 20 in each of grades K, 1, 2, and 3 were used as subjects. Utilizing sound experimental procedures, ANOVA procedures detected differences due to grade and type of matrix with a significant interaction occurring. No differences occurred between grade 2 and grade 3 in a post-hoc analysis for any matrix type. For these two grades, the mean scores were approximately 2.50 (out of 3) except for the DD matrix, where the mean scores were approximately 1.80 (out of 3). The kindergarteners differed from the second and third graders on all matrix types where the mean score for kindergarteners was about 41 percent across matrix types. Kindergarteners differed from first graders on all matrix types except for the CC and DD matrices, and first graders differed from second and third graders on all matrix types except DD. The mean score across matrix types for grade 1 was approximately 57 percent, for grade 2 approximately 74 percent, and for grade 3 approximately 79 percent. There were significantly more correct responses to CC matrices (mean 2.16) than to FF matrices (mean 1.82), and significantly more correct responses to FF matrices than to DD matrices (mean 1.54).

The results of the above two studies, when considered conjunctively, clarify modes of classification of children of various ages and types of collections formed through classification. Untrained five and six year old children perform quite modestly on double matrix tasks even when concrete content is used. This result coupled with the fact that children in the same age range form, with high incidence,

local collections based on a perceptible or thematic mode of classification supports the developmental theory of Inhelder and Piaget (1969) as it concerns classification.

Classification In Elementary School Mathematics

The ability to classify in any given situation presupposes criteria for classification. Classification becomes logical only if there is an explicit recognition of the criteria used to classify. As already noted, such criteria may be based on perceptible aspects of objects, on functional aspects of objects, or on designative aspects of objects. All such criteria may or may not entail logical classification. That is, one cannot tell whether a child forms logical classes by considering only the mode of classification. For example, a child could use a functional mode of classification but change criteria each time a new object is considered in such a way that logical classes are never formed.

Class Inclusion. The relation of class inclusion is implicit in rational counting procedures. If a collection of objects are counted, then the collection containing, say, four objects must be considered as being part of a set of five objects as well as a collection in itself, so that five is one more than four and four is one fewer than five. A child, given four circles and one rectangle, may be able to point to each circle, each shape, and the rectangle, but still insist there are more circles than shapes. Essentially, such a child may be able to state the criteria of each class but fail to handle the inclusion relationship between the two. Inhelder and Piaget (1969, p. 7) ask that a child be able to give an intensive definition of a class in terms of a more general class as well as handle the

inclusion relation before operational existence of classes is admitted on the part of the child. The intension of a class is just the set of properties common to the members of that class together with the differences that distinguishes them from another class. Not only, then, is it necessary that explicit recognition of the criteria of classification exist for classification to be logical, but it is also necessary that the class inclusion relation be handled by the child.

An essential aspect of class inclusion is partial ordering. In the Grouping Structure concerned with cognition of simple hierarchies of classes, which involves class inclusion, Piaget (1950, p. 43) gives, embedded in a zoological classification, a statement analogous to the following:

(1) Combinativity: $A \cup A^1 = B$

(2) Reversibility: If $A \cup A^1 = B$, then $A = B - A^1$

(3) Associativity: $(A \cup A^1) \cup B^1 = A \cup (A^1 \cup B^1)$

(4) General Operation of Identity: $A \cup \emptyset = A$

(5) Special Identities: (a) $A \cup A = A$, (b) $A \cup B = B$ where $A \subset B$

This grouping describes essential operations and relations involved in cognition of simple hierarchies of classes. Proficiency with the class inclusion relation involves two capabilities. First is the capability to combine classes (combinativity) and decompose classes (reversibility). Second is the capability to hold in mind a total class and its subclasses at the same time, made possible through combinativity and reversibility. Inhelder and Piaget (1969) reported various interviews of children surrounding class inclusion and stated in their conclusions that ". . .it is one thing to carry out the union . . . and quite another to understand that it is logically equivalent to its

inverse . . ., which means that the whole . . . retains its . . . identity . . . . The conservation of the whole and the quantitative comparison of whole and part are the two essential characteristics of genuine class inclusion (p. 117)." It is not until approximately eight years of age (Inhelder and Piaget, 1969, p. 109) that a majority of children obtain genuine class-inclusion.

D. Johnson (1971) in a comprehensive study concerning classification, investigated effects of classificational experiences on the performance of kindergarten and first grade children on the class-inclusion problem. Grade and IQ (80-100; 105-125) were used as classificational variables. Twenty children were assigned to each of the experimental and control groups within each IQ level and Grade, for a total of 80 subjects, forty experimental and forty control.

A total of 16 class-inclusion items were constructed. The content of eight of the items were what Parker and Halbrook (1969) categorized as Designative, the content of four was concrete, and the content of four was a mixture of designative and functional. The items varied in other ways such as in the inclusion of extraneous objects, in the existence of three subcategories instead of two, and in randomly assorted arrangements. An example item was where there were five red flowers, four yellow flowers, and two American flags, with the flowers arranged at random. The questions asked were, "Are there more flowers than red flowers," and "Are there more red flowers than flowers," <u>after</u> the child had pointed to each flower, each red flower, each yellow flower, and to each flag. To be successful on the item, the child had to answer each question correctly.

Seventeen experimental sessions were held. Thirteen of the sessions dealt directly with classification and four dealt with the

relations "more than," "fewer than," and "as many as." All of the content categories; concrete, functional, and designative were used in the sessions with a predominance of concrete content. The first three sessions dealt with categorization based on the relations "same color as", "same shape as", the criteria of being red or not red (or other colors), things to ride in, things to eat with, and animals; incorporating the terms "all", "some", and "not" (set complementation). The next three sessions dealt with set intersection in the context of multiple attributes of objects. For example, triangles would be sorted into one hula hoop and red objects into another hula hoop, so the problem would arise as to where the red triangles should go. The last two sessions were practice sessions in the formation of various classes.

Results of the experimental session were negligible on the class-inclusion test. The KR-20 reliability was .75 but the overall mean was only approximately 29 percent (D. Johnson, 1971, p. 98). Effects due to Grade was not significant but effects due to IQ were, with the means 20 and 38 percent for the lower and higher categories, respectively. No differences in over-all item difficulty was observed between the items involving concrete content, designative content, and a mixture of designative and functional content. It appears, then, that the class-inclusion relation is resistent to classificational instruction except for didactic methods such as that given by Kohnstamm (1967). The results of his training methods, however, have come under question concerning the deepness of the learning (Inhelder and Sinclair, 1969).

When considering the results of D. Johnson's (1971) study and the way in which addition and subtraction are presented in school mathematics curricula, logical inconsistencies are present. The class

inclusion relation, as well as explicit recognition of criteria for classification, is inextricably involved in addition and subtraction of whole numbers. While the following study is not directed explicitly toward investigation of class inclusion and addition and subtraction, it points out a need for such investigations to be conducted.

In a study concerned with problem solving performances of first grade children, Steffe and Johnson (1971) presented end-of-the-year first grade children with four problem types; a + b = n, a - b = n, a + n = b, n + a = b under two problem conditions; manipulatable objects present and manipulatable objects absent. The mean scores for the problem types were given for each problem condition to the nearest percent and are quoted in Table 1.

TABLE 1

	Objects Present	Objects Absent
a + b = n	82	64
a - b = n	59	38
a + n = b	65	38
n + a = b	61	37

Twelve problems were administered within each category, ten of which involved designative content and two concrete content. An example of a problem is, "There are eight animals in a corral. Five of these animals are horses and the other animals are cows. How many animals are cows?" To solve this problem, it would seem essential to conceive

of the animals as made up of horses and cows, and the animals, disregarding the horses, as cows. That is, it would seem that class inclusion is involved in the solution. In view of the low mean scores on the problem solving tests and on the class inclusion test, studies designed to investigate relationships between performance on the class-inclusion problem and addition and subtraction are warrented. Such studies should take into account various types and modes of classification and problem conditions.

Multiple Classification. Multiple classificational ability, of which class inclusion is an example, is logically related to successful performance in early elementary school mathematics. That is, an ability to conceive of an object as belonging to two or more classes at the same time is fundamental. In the same study as mentioned earlier, D. Johnson (1971) also constructed collections of intersecting ring items and matrix items (two by two and three by three). All of the items of these two item sets could be justifiably classified as utilizing concrete content. The treatment was highly effective in improving mean performances on these two measures. The mean score on the intersecting ring items was 44 percent for the experimentals and 12 percent for the controls with no grade differences reported. On the matrix items combined, the mean score was 59 percent for the experimentals and 41 percent for the controls. With respect to matrix items, the results of D. Johnson's (1971) experiment are consistent with an experiment reported by Siegel and Kresh (1970). These latter authors utilized two basic variations of the matrix task; an incomplete matrix task and an attribute cell task. In an incomplete matrix (3 x 3) task, some matrix entries were left empty and a

child was to fill those missing from a selection. In the attribute cell task, the attributes of the rows and columns were given but no entries. Without going into a full review of the study, the authors performed a behavioral analysis of the two matrix tasks and, based on that analysis, succeeded in training to criterion 80 percent of the kindergarten children in the sample after only 10 trials on the incomplete matrix task. The authors interpret their data as demonstrating that pre-operational children can learn concrete operational skills through practice on a series of simpler tasks (Siegel and Kresh, 1970, p. 68).

The results of both D. Johnson's and Siegel and Kresh's experiments must be tempered by the following analysis. Piaget (1964) distinguishes clearly between physical and logical mathematical experience. Physical experience ". . . consists of acting upon objects and drawing some knowledge about the objects by abstraction from the objects (Piaget, 1964, p. 11)." An example is given concerning the comparative weight of two objects. A child could discover which of two or more objects is the heavier one by weighing them. Logical mathematical experience is experience where ". . . knowledge is not drawn from the objects, but it is drawn by the actions effected upon the objects (Piaget, 1964, p. 12)." The example Piaget gives is a case where a child counted a row of pebbles from each end and discovered that it didn't make any difference which end he had started from, but he got ten in each case. In case of knowledge of the physical type, it is possible to show a child that he is not correct whereas in logical-mathematical experience it is not possible to demonstrate to the child that he is wrong. Sinclair (1971, p. 55) uses an example of a class-inclusion problem using designative content when illustrating such an impossibility. However, in case of Siegel

and Kresh's study concrete content was utilized so that it was possible for the child to be corrected when he was incorrect. For example, in the attribute cell task, a matrix cell could be filled by physically checking if the element matched the respective row attribute and the respective column attribute. A strong hypothesis exists, then, that the learned behavior exhibited by the children in both types of multiple classification tasks was in the realm of physical knowledge. Because of the lack of a variety of multiple classificational tasks, no information is available concerning the scope or depth of the induced behavior. Herein lies an area of fruitful investigation.

In D. Johnson's study, a larger variety of multiplication of classes tasks (intersecting ring and three by three and two by two incomplete matrix tasks) were included than was included by Siegel and Kresh, but all involved concrete content. Thereby, the question also remains whether D. Johnson induced anything beyond physical knowledge (or regards classification) in the children in his sample.

<u>Relations and Classification</u>. Classification based on concrete, functional, and designative content is especially important in work with number in the elementary school. Other criteria for classification are also important in the area of length as well as in the area of number. Equivalence relations and their accompanying difference relations are seldom studied in relation to classification. That relations are overlooked in studies on classification seems a bit puzzling because for each partition of a collection of objects, an equivalence and difference relation are determined and vice versa. That is, if $\{P_1, P_2\ P_3, \ldots, P_n\}$ forms a partition of a collection A, then an equivalence relation "$\sim$" has been determined and is defined as

follows. If x,y are elements of A, then $x \sim y$ if and only if $x \in Pi$ and $y \in Pi$ for some i. Conversely, given an equivalence relation "$\sim$" defined on a collection A, the relation determines a partition of A. For a partition to be formed, the properties of the relation must be employed. As an example, consider the case where a child is given a collection of "rods" and asked to put the rods into piles so that the rods in each pile are the same length. It would seem essential for a child to use properties of the equivalence relation "same length as" in order to perform the classification. For if any rod s is taken from the collection, then s is either classified with itself (reflexivity) or there is another rod r the same length as s. An exhaustive sorting must take place in order for a partition to be formed. Not only must the sorting be exhaustive, but any given subcollection must contain all of the sticks the same length as any given stick of that subcollection. In this latter case, the sorting is called completely exhaustive. For a sorting of sticks to be completely exhaustive, a child would have to use the properties of the relation, at least as follows. Say a child has placed two sticks, s and r, into a given pile and finds another stick t where t is the same length as s. The child must then realize that t is the same length as r for the classification to be what Olver and Hornsby termed "superordinate." It is definitely a viable hypothesis that children who form complexive collections as defined by Olver and Hornsby are not operational as regards transitivity.

A rather extensive behavioral analysis (see appendix) of the capability to determine the inner and outer measure of a segment was conducted by the author (Steffe, et. al, 1970). While this behavioral analysis has not been completely psychometrically validated,

M. Johnson (1971) has conducted a study concerned in part with classificational aspects of the hierarchy. The basic logic of the classificational aspect of M. Johnson's (1971) study was to train children in the formulation of equivalence classes using length relations (M_{41}) and then investigate relationships of classificational ability and conservation of length relations (M_{22}) and transitivity (M_{34}). It is known that transitivity of length relations and conservation of length relations are hierarchically related (Divers, 1970).

M. Johnson (1971) used 39 first grade children and 42 second grade children as subjects. All children received a pretreatment designed to acquaint them with the relations. They then were randomly assigned to treatment and control groups. The treatment group received experiences in classification (M_{41}) and seriation (M_{61}) where two sessions were devoted to classification. At the end of the treatment session, as part of the testing program, a classification test, conservation of "same length as" test, and transitivity of "same length as" test were administered. The classification test consisted of an active and a passive condition. The active condition consisted of (1) a situation where the children were asked to sort a collection of sticks into three collections given three sticks on which to base the sorting, and (2) a situation where children were given a collection of sticks and instructed to put all of the sticks together that belonged together. The passive condition consisted of a situation where sticks were completely sorted and the child was asked to find out why the sticks were piled up in the particular way given. In the case of active condition (1), 29 of the experimental children and 33 of the control children made completely exhaustive classifications. In the case of the passive condition, 29 experimental children and 30 control children

did not discover the criteria of classification and 11 experimental and 10 control children showed evidence of awareness of the criteria for classification. In the case of active condition (2), only five experimental and one control subject made no attempt to group the sticks. Seventeen experimental and 21 control subjects made some piles, but did not make completely exhaustive sortings. Nineteen experimental and 18 control subjects made completely exhaustive sortings except for possibly one stick.

It is apparent from the above data that the training given on classification in no way improved classificational performance of the children. While the performance of both the experimental and control groups was quite good when the criteria was given (the item was just a sorting item!), the performances were not as good when the children had to abstract the criteria for classification. Here, the classificational performance was poor in the passive condition but somewhat better in the active condition. Yet, in the latter condition 44 children out of 81 did not show evidence of classification ability based on "same length as." These data illustrate quite lucidly the difference between physical knowledge and logical-mathematical knowledge. When the criteria was given to the child (as well as standard sticks) on which to base the sorting, physical knowledge was all that was required to solve the problem. However, when the criteria was not given, the child was required to abstract the criteria for classification and the task required logical mathematical knowledge as well as physical knowledge.

It would seem that the classification task in the active condition for which the criteria was provided would not be at all related to transitivity performance on "same length as." Data reported

by M. Johnson (1971, p. 89) supports this contention. The data do not support the contention, however, that transitivity of "same length as" is related to classification performance on tasks for which the criteria was not given. This latter result is not at all consistent with the theoretical analysis given and demands further research utilizing less structured tasks of transitivity to ensure operational transitivity is being assessed.

The combining of classification and seriation and multiple seriation becomes especially important in M_{71} and beyond in the behavioral analysis. In the case of M_{72}, various examples have been given (Steffe, 1971, p. 336) and are repeated here for clarification. If a child is asked to compare the length of two polygonal paths P_1 and P_2 each of which consist of congruent segments, a comparison may or may not be logically possible based on the information available. If the segments of P_1 are congruent to the segments of P_2, a comparison is always possible. However, if the segments of P_1 are, say, shorter than the segments of P_2, a comparison may or may not be possible depending on how many segments are in P_1 and in P_2. In any comparison of P_1 and in P_2, a child must coordinate two items of information; how the segments of P_1 are related to the segments of P_2 in terms of length and number. Evidence does exist on children's ability to make such coordinations. M. Johnson (1971) actually conducted a sequence of two experiments the second of which has been discussed in part. In the first experiment, six items were constructed designed to measure the ability to combine relations and combine classification and relations. Three items were in a 2 x 2 matrix format and three were in a 3 x 3 matrix format. Two items were designed to

measure a coordination of classification and relations, one 2 x 2 matrix item and one 3 x 3 matrix item. The columns were determined by the numbers "one," "two," or "three" and the rows by three inch sticks, four inch sticks, or five inch sticks. Obviously, these two items could also be considered as measuring the combination of relations and are directly related to the abilities outlined in M_{72}. Overall item difficulties were .14 and .26 for these two items, respectively (.14 means 14 percent of the children scored the item correctly) for kindergarten, first, and second grade children. These item difficulties were quite comparable to the other four items which involved only length relations. No important differences existed due to grade nor treatment on this test (M. Johnson, 1971), where the treatment involved experiences in multiple seriation and a combination of classification and seriation.

The results of M. Johnson's study, when contrasted with the results of D. Johnson's and Siegel and Kresh's studies give a broader perspective of children's ability to handle matrix items and confirms the importance of distinguishing between the two poles of knowledge outlined by Piaget.

An Overview

It is apparent that while a great deal is known about classificational abilities of young children, more exploration is needed. As an example supplementary to those already suggested, consider the low scores observed by M. Johnson (1971) on multiple relational and/or classificational tasks by kindergarten, first, and second grade children. These tasks ought to be in the scope of tasks solvable by children in the stage of concrete operations. Piaget's (Flavell, 1965) Grouping Structures concerning multiplication of classes and

relations are supposedly elaborated at approximately eight years of age with seven year old children at least in a transitional stage. Yet, M. Johnson observed no differences between kindergarten and second grade children on the tasks under discussion. Such a result is surprising and demands further investigation especially in light of the logical dependency of measurement processes on the ability to solve multiple classificational or relational items, a dependency which needs investigation.

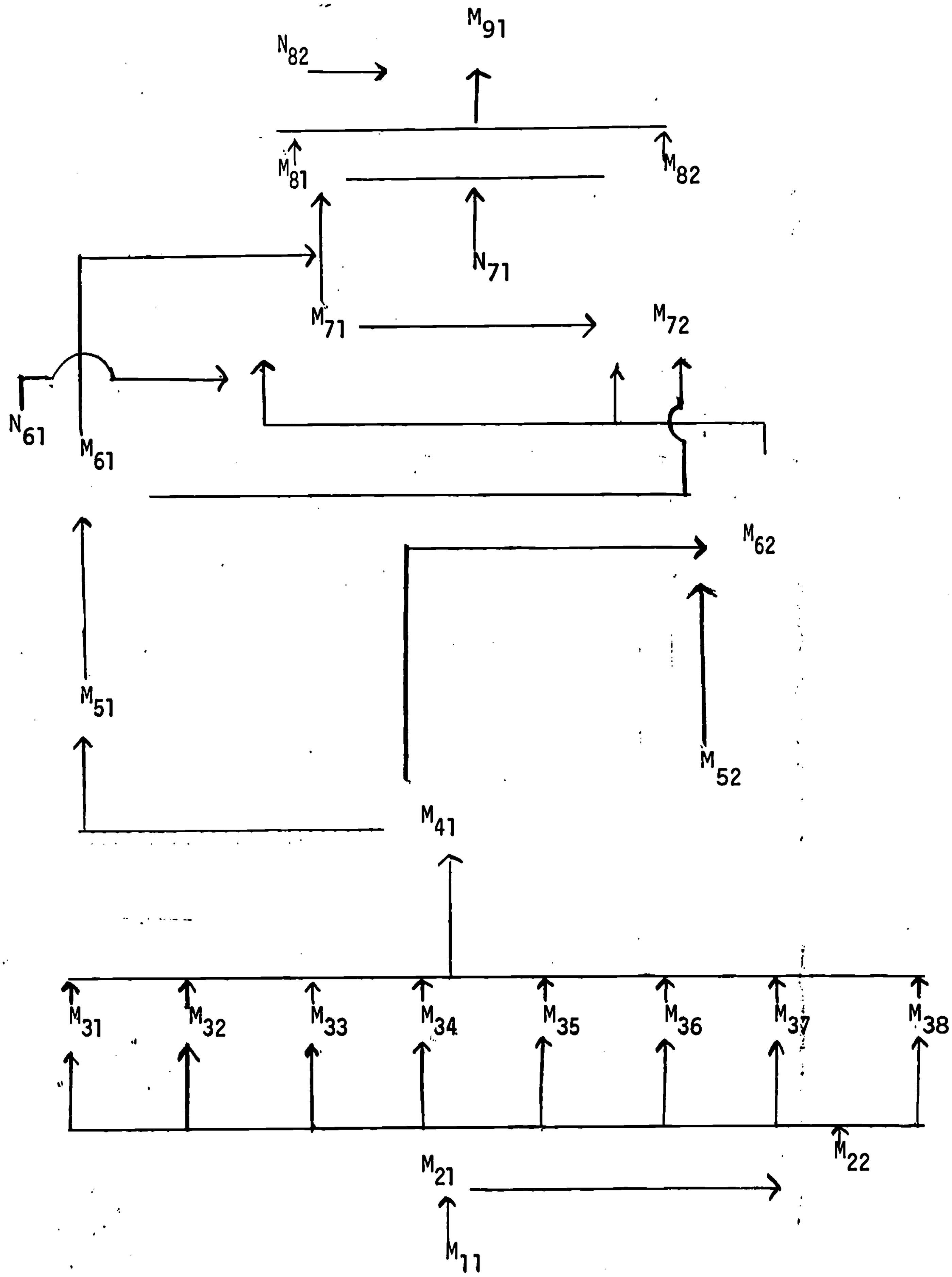
The matrix items used by M. Johnson differed from those used by D. Johnson, Siegel and Kresh, and Parker and Halbrook in the content used. While it would be possible to classify M. Johnson's items under designative content (number and length), their content at this time does seem to be different enough from the designative content used by Parker and Halbrook to warrent special consideration due to the importance of length and number in mathematics. At present, then, it seems appropriate to construct a variety of multiple classificational and relational items (class inclusion, matrix items, intersecting ring items) across a variety of content in order to investigate more carefully convergence in logical thinking on those items under various experimental conditions.

Because of the logical relationship between classification and relations, convergence in logical thinking concerning classification and relational properties needs investigation. As noted, preliminary results by M. Johnson are not consistent with logical interdependencies of transitivity of "same length as" and completely exhaustive sortings of "rods" based on "same length as." It is possible to investigate such hypothesized interdependencies in other contexts. As an extreme example, consider the child who is classifying a collection of objects

on the basis of "things to ride in," a functional classification. Imagine the child selects a model car and classifies it as something to ride in, say A. If he also views another object B as something to ride in and agrees that A is like B because they are both something to ride in and if he also agrees that B is like C for the same reason, will he declare that A is like C? In short, if the child forms complexive collections rather than superordinate collections, it is reasonable to hypothesize that that child cannot yet use properties of equivalence relations. Such investigations remain to be conducted.

As intrinsically interesting as classificational behavior may be, it should be studied in relation to other important developmental phenomena in mathematics, especially number and length. An area of study has been already suggested in that relations between class-inclusion and addition and subtraction needs more investigation. Other than this specific problem, the author (Steffe, et.al., 1970) has also conducted a behavioral analysis involving number. While a host of studies (Carey and Steffe, 1968; Divers, 1970; Owens and Steffe, In Press; Johnson, D., 1971; Owens, 1972; and Steffe and Carey, 1972) have already been conducted surrounding this analysis, more remain to be completed than have been done.

APPENDIX



M_{91}
N_{82}
M_{81}
M_{82}
N_{71}
M_{71}
M_{72}
N_{61}
M_{61}
M_{62}
M_{51}
M_{52}
M_{41}
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M_{35}
M_{36}
M_{37}
M_{38}
M_{22}
M_{21}
M_{11}

In the following definitions, A, B, and C represent open curves of finite length or simple closed curves, and T represents any length-preserving transformation. A curve and a physical representation of its trace will not be distinguished. The possibility of "straightening" an open curve will be assumed.

M_{11} Associates a name with a curve:

A child is said to be able to associate a name with a curve if, and only if, when shown a curve, he names it.

M_{21} Establishes a length relation between two open curves:

Given two curves A and B, a child is said to be able to establish a length relation "*" ("longer than," "shorter than," or the "same length as") between A and B if, and only if, he

(1) places each curve on a line in such a way that two endpoints (left or right) coincide;

(2) compares the relative positions of the two remaining endpoints; and then,

(3) on the basis of (a) and (b), deduces that A*B, if in fact it is true that A*B.

M_{22} Conserves a length relation between two open curves:

A length relation between two curves, A and B, is conserved by a child if, and only if, the relation is

(1) established by the child; and,

(2) retained, regardless of any relation-preserving transformation on one or both of the curves.

M_{31} Uses consequences of length relations - A shorter (longer) than B is equivalent to B longer (shorter) than A:

A child is said to be able to use the above consequence if, and only if, from establishing A shorter (longer) than B, he is able to deduce that B is longer (shorter) than A;

regardless of any relation preserving transformation on one or both of the curves.

M_{32} Uses consequences of length relations - to say that A is the same length as B implies that A is not shorter (longer) than B:

A child is said to be able to use the above consequence if, and only if, from establishing A the same length as B, he is able to deduce that A is not shorter (longer) than B, regardless of any relation-preserving transformation on one or both of the curves.

M_{34} Uses transitive property of "the same length as," "longer than," or "shorter than":

A child is said to be able to use the transitive property of "*" if, and only if, from establishing that A*B and B*C he is able to deduce that A*C, regardless of any relation-preserving transformation on one or both of the curves.

M_{35} Uses symmetric property of "the same length as":

A child is said to be able to use the symmetric property of " " if, and only if, from establishing A B, he is able to deduce that B A, regardless of any relation-preserving transformation on one or both of the curves.

M_{36} Uses asymmetric property of "shorter than" and "longer than":

A child is said to be able to use the asymmetric property of "$\propto$" if, and only if, from establishing $A \propto B$, he is able to deduce that $B \not\propto A$, regardless of any relation-preserving transformation on one or both of the curves.

M_{37} Uses reflexive property of "the same length as":

A child is said to be able to use the reflexive property of "$\sim$" if, and only if, he is able to deduce that $A \sim T(A)$, where T is a length-preserving transformation.

M_{38} Uses nonreflexive property of "longer than" or "shorter than":

A child is said to be able to use the nonreflexive property of "$\propto$" if, and only if, he is able to deduce that $A \not\propto T(A)$, where T is a length-preserving transformation.

M_{41} Forms equivalence classes using length relations:

A child is said to be able to form an equivalence class of curves if, and only if, given a collection of open curves he is able to:

(1) partition the collection of curves into equivalence sets by selecting a curve and then selecting those curves of the same length as the selected curve and proceeding in this way until the collection is exhausted;

(2) deduce that any two curves in an equivalence set are of the same length and not of different lengths; and,

(3) deduce that for any two curves A and B in different equivalence sets, A is of different length than B.

M_{51} Uses consequence of length relations $A \sim B$ and $B \not\sim C$ implies $A \not\sim C$:

A child is said to be able to use the above consequence if, and only if, from establishing $A \sim B$ and $B \not\sim C$, he is able to deduce that $A \not\sim C$ and that no other relation obtains, regardless of any relation-preserving transformation on one or both of the curves.

M_{52} Given K open curves, forms a path:

A child is said to be able to form a path P from a collection of K open curves if, and only if, he places the curves with endpoints coinciding with the endpoints of no more than two curves coincident.

M_{61} Orders a set of curves:

A child is said to be able to order a set of curves according to "$\trianglelefteq$" if, and only if, he can find a curve which is "shorter than" or the "same length as" any other curve, and then a curve which is "shorter than" or the "same length as" any remaining curve, etc. Similarly for "longer than" or "the same length as."

M_{62} Given K segments of an equivalence class E of segments, forms a polygonal path:

A child is said to be able to form a polygonal path P from a subcollection $E_K = \{e_1, e_2, \ldots, e_K\}$ of segments of E if, and only if, he places the segments with endpoints coinciding with no more than the endpoints of two segments coincident.

M_{71} Determines the length relation between two paths P_K and P_j from the matching relation between E_K and E_j; where E_K and E_j are disjoint subsets of the same equivalence class of curves and P_i is the path formed from the curves of E_i:

A child is said to be able to determine the length relation between P_K and P_j from the relation between E_K and E_j if, and only if, he is able to;

(1) establish a relation between E_K and E_j;

(2) form the paths P_K and P_j; and then,

(3) on the basis of only (1) and (2) deduces the length relation between P_K and P_j.

M_{72} Determines a length relation between two paths P_{E_K} and P_{F_K} from a corresponding length relation between two open curves e_K and f_K, where $e_K \in E_K$ and $f_K \in F_K$, which are subsets of two equivalence classes of open curves E and F, respectively. P_{E_K} and P_{F_K} are paths formed from the open curves of E_K and F_K, respectively:

A child is said to be able to determine, if possible, a length relation between two paths P_{E_K} and P_{F_K} from a corresponding length relation between e_K and f_K if, and only if, he establishes:

(1) how e_K is related to f_K;

(2) that the e_i are equivalent and the f_i are equivalent (the same length);

(3) how E_K is related to F_K, and then

(4) on the basis of (1), (2), and (3) concludes how P_{E_K} is related to P_{F_K} if it is possible to relate them.

M_{81} Orders a sequence of paths from the ordering of a sequence of sets of open curves from which the paths were formed:

Given a sequence $E_1, E_2, \ldots, E_n$ of disjoint subcollections of an equivalence class E, and a sequence of paths P_1, $P_2, \ldots, P_n$ corresponding to the E_i's, a child is said to be able to determine the order of the sequence of paths $P_1 \underline{\propto} P_2 \underline{\propto} \ldots \underline{\propto} P_n$
($\underline{\propto}$: shorter than or the same length as) from the order of the sequence of sets $E_1 \underline{\propto} E_2 \underline{\propto} \ldots \underline{\propto} E_n$ ($\underline{\propto}$: fewer than or

as many as) if, and only if, he is able to arrange the paths into order on the basis of having ordered the sets from which the paths were formed.

M_{82} Orders a sequence of paths from ordering subparts of the paths:

Given a sequence E_1, $E_2, \ldots, E_n$ of finite equivalence sets of curves (all the same number of elements) and P_1, $P_2, \ldots, P_n$ the respective paths (of finite length), a child is said to be able to determine the order $P_1 \underline{\alpha} P_2 \underline{\alpha} P_3 \cdots \underline{\alpha} P_n$ from $e_1 \underline{\alpha} e_2 \underline{\alpha} e_3 \cdots \underline{\alpha} e_n$ if, and only if, he arranges the paths into order on the basis of having ordered representative curves from which the paths were formed.

M_{91} Determines inner and outer measure of a segment P:

A child is said to be able to determine the inner and outer measure of a segment P if, and only if, he;

(1) forms two polygonal paths from the segments of an equivalence class in such a way that the two polygonal paths are one segment apart in length and such that P is longer than one segment and shorter than the other; and,

(2) counts the curves in each polygonal path. In some cases, one polygonal path may suffice.

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